Small open economy models

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This note studies a class of models in which an economy has access to international borrowing and lending. We say the economy is small because it takes the interest rate and the prices of tradeable goods as given — though not necessarily constant. A major focus of this literature is the behavior of the current account, which summarizes the borrowing and lending done in international markets.

We will begin with exchange models which are easier to characterize than models with production. First, we will tackle perfect foresight and then models with stochastic endowments. The goal is to understand how agents use external debt to smooth consumption and what this implies for the behavior of the current account and other variables of interest. We will then move on to models with production, which will usually require us to solve the models numerically.

1.1 Perfect foresight exchange economy

The country is populated by a representative consumer with preferences over sequences of consumption

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{1}$$

where u is a well-behaved utility function and $\beta < 1$ is the discount factor. The household faces a sequence of budget constraints of the form

$$c_t + b_{t+1} \leq \left(1+r\right) b_t + y_t \qquad t = 0, 1, \ldots$$

with b_0 given and $c_t > 0$. The household is endowed with tradeable goods y_t. In this section we assume that the sequence of y is known. The beginning-of-period holdings of one-period bonds are b_t. Our notation implies that when $b_t < 0$ the household is a net borrower from the rest of the world.

This framework allows us to define some often referenced concepts. The *trade balance* or *net exports* is

$$tb_t = y_t - c_t \tag{2}$$

[What does $tb_t < 0$ mean? $tb_t > 0$?] When there is only one good in the economy, there cannot be both imports and exports in the economy. The *current account* is the change in the country's net foreign asset position,

$$\begin{aligned} ca_t &= b_{t+1} - b_t && (3)\\ &= y_t - c_t + r b_t && (4)\\ &= tb_t + r b_t && (5) \end{aligned}$$

[To get to the second equality, solve the budget constraint for b_{t+1}.] The current account is the trade balance plus net capital income (rb_t). This definition implies that unbalanced trade ($tb_t \neq 0$) is associated with a change in the net foreign asset position: either the country is borrowing from or lending to the rest of the world.

Our goal is to work out the consumption behavior of the household. It is easiest to do so with the lifetime budget constraint. The idea is to deflate the sequence budget constraint at t by $(1+r)^t$ and then add them up. The first two terms are

$$c_0 + b_1 = (1+r)b_0 + y_0 \quad (6)$$
$$\frac{c_1 + b_2}{1+r} = \frac{(1+r)b_1 + y_1}{1+r} \quad (7)$$

Add these two equations and find

$$c_0 + \frac{c_1}{1+r} + \frac{b_2}{1+r} = (1+r)b_0 + y_0 + \frac{y_1}{1+r}. \quad (8)$$

Notice that b_1 cancels out and the beginning of a discounted sum of consumption and income are forming. If we continue in this way we find

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} + \lim_{t\to\infty} \frac{b_{t+1}}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} + (1+r)b_0. \quad (9)$$

The transversality condition says that $\lim_{t\to\infty} \frac{b_{t+1}}{(1+r)^t} = 0$, and we have that the discounted value of consumption is equal to the households wealth, which is the discounted value of the endowment plus the beginning net foreign asset position.

We will maximize (1) subject to (9). The Lagrangian is

$$\Lambda = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left(b_0(1+r) + \sum_{t=0}^{\infty} \frac{y_t - c_t}{(1+r)^t} \right) \quad (10)$$

and the necessary first-order condition with respect to c_t is

$$\beta^t u'(c_t) = \frac{\lambda}{(1+r)^t} \quad (11)$$
$$u'(c_t) = \frac{\lambda}{[\beta(1+r)]^t} \quad (12)$$

The solution sets the marginal utility of consumption equal to the marginal utility of a unit of wealth (λ) discounted by both the consumer's discount factor and the gross interest rate, which is the rate in which wealth grows if carried over a period.

Consumption behavior depends on $\beta(1+r)$. Assuming u is well behaved, there are three possibilities

1. $\beta(1+r) > 1$ implies that $c_t \to \infty$. Relative to the rate of return, agents are patient. Accumulate assets and consume a lot in the future.
2. $\beta(1+r) = 1$ implies that $c_t = \bar{c}$. The household's patience exactly cancels out the return on saving.
3. $\beta(1+r) < 1$ implies that $c_t \to 0$. Agents are impatient relative to the rate of return, so they consume early.

If we assume that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ we can solve for λ,

$$
\begin{aligned}
c_t^{-\sigma} &= \frac{\lambda}{[\beta(1+r)]^t} && (13)\\
c_t &= \lambda^{\frac{-1}{\sigma}}[\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}]^t && (14)\\
\sum \frac{c_t}{(1+r)^t} &= \lambda^{\frac{-1}{\sigma}} \sum \frac{[\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}]^t}{(1+r)^t} && (15)\\
\sum \frac{y_t}{(1+r)^t} + (1+r)b_0 &= \lambda^{\frac{-1}{\sigma}} \sum \frac{[\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}]^t}{(1+r)^t} && (16)\\
\sum \frac{y_t}{(1+r)^t} + (1+r)b_0 &= \lambda^{\frac{-1}{\sigma}} \sum [\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}}]^t && (17)\\
Y &= \lambda^{\frac{-1}{\sigma}} \sum [\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}}]^t && (18)\\
\frac{Y}{\sum [\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}}]^t} &= \lambda^{\frac{-1}{\sigma}} && (19)\\
Y^{-\sigma}\left(\sum [\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}}]^t\right)^{\sigma} &= \lambda && (20)
\end{aligned}
$$

If Y is larger, the marginal value of wealth falls; The greater is σ the greater is this effect. [As σ increases, the agent is more risk averse.] We need $\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}} < 1$ for a solution to exist. Consumption is

$$
c_t = Y \frac{[\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}]^t}{\sum [\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}}]^t} \tag{21}
$$

Suppose $\beta(1+r) = 1$, so $\beta = 1/(1+r)$. The numerator on the right hand side is one and the denominator is $1/(r/(1+r))$, so consumption is $c_t = \frac{r}{1+r}Y$. Notice that consumption is constant, which we already knew.

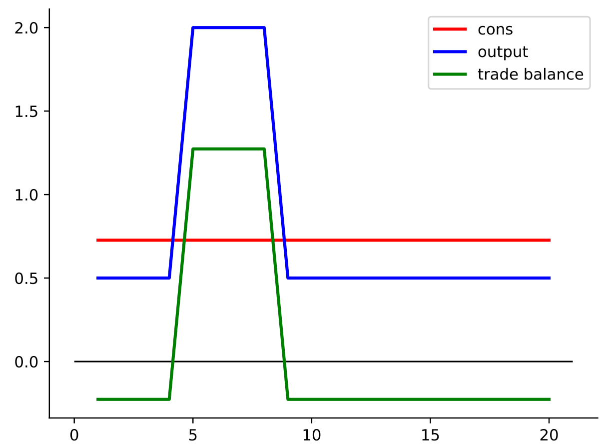
Now we know about consumption. What does the path of consumption imply for borrowing and lending? Iterating backwards, the bond policy function is

$$
\begin{aligned}
b_{t+1} &= tb_t + (1+r)b_t && (22)\\
&= \sum_{s=0}^{t} tb_s(1+r)^{t-s} + (1+r)^t b_0 && (23)\\
&= (1+r)^t b_0 + \sum_{s=0}^{t} (y_s - c_s)(1+r)^{t-s} && (24)
\end{aligned}
$$

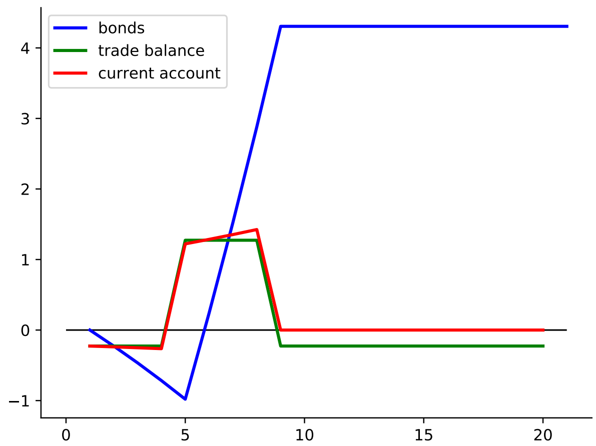
which says that today's net foreign asset position is the discounted value of the initial position, plus the discounted value of all the previous trade balances, which keep track of the borrowing and lending done in the past. [Note that the trade balance plays the same role as the primary surplus in a fiscal policy setting.] We can also iterate forward on the budget constraint to find

$$
\begin{aligned}
b_{t+1} &= \frac{b_{t+2}}{(1+r)} - \frac{tb_{t+1}}{(1+r)} && (25)\\
&= \lim_{s\to\infty} \frac{b_{t+s}}{(1+r)^{t+s-1}} - \sum_{s=0}^{\infty} \frac{tb_{t+s}}{(1+r)^{t+s}}. && (26)
\end{aligned}
$$

Figure 1: Deterministic fluctuation in endowment



(a) Output and consumption



(b) Current account and net foreign assets

This shows us that today's net foreign asset position (as long as there is no default) summarizes both the past and future borrowing and lending. For example: If the net foreign asset position is negative, it means that the country has consumed more than it was endowed in the past. It also means that the country must run trade surpluses in the future ($tb > 0$).

The program `soe-deterministic.jl` computes an example in which output is constant at y_L except for a brief period in which output is $y_H > y_L$. The results are plotted in Figure 1. When $(1+r)\beta = 1$ consumption will be constant and at a level above y_L. When the endowment is y_L, the trade balance is negative and net foreign asset position decreases. When the endowment switches to y_H, the trade balance is positive and the net foreign asset position becomes positive. Notice the difference between the current account and the trade balance after the endowment returns to y_L. The country finances its trade deficit with the return on its asset holdings.

1.2 Stochastic exchange economy

In this section we introduce uncertainty into our endowment economy. There are an infinite number of periods indexed by t. In each period an event s_t occurs. The set of possible events at t is S_t. Let the history of events that have occurred up to period t be denoted $s^t = (s_0, s_1, \dots, s_t)$ and the probability that we have reached this history be $\pi(s^t)$. The set of histories that can occur at time t is S^t. [If this notation confuses you, draw out the event tree. A history is one particular path (branch) from date 0 through date t.]

The uncertainty is over the endowment, $y(s^t)$. We will assume that the endowment follows an autoregressive process,

$$y(s^t) \quad = \quad \rho y(s^{t-1}) + \epsilon(s^t) \tag{27}$$

where $\rho \in (0, 1)$ and ϵ_t is an innovation process that is i.i.d. over time. [I guess we need that

$y(s^{t-1})$ is large enough that we never have negative endowment?[1]] To make our lives simple, let $u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$. The household's problem is now

$$\max_{c(s^t), b(s^{t+1})} \quad -\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \pi(s^t)\beta^t \frac{1}{2}\left(c(s^t) - \bar{c}\right)^2 \tag{28}$$

$$\text{s.t.} \;\; c(s^t) + b(s^{t+1}) \leq (1+r)b(s^t) + y(s^t) \qquad \forall s^t \in S^t \tag{29}$$

with b_0 given and $\bar{c} \geq c(s^t) \geq 0$. The first order conditions are, for each s^t,

$$-\pi(s^t)\beta^t\left(c(s^t) - \bar{c}\right) = \lambda(s^t) \tag{30}$$

$$\lambda(s^t) = (1+r)\sum_{s^{t+1}} \lambda(s^{t+1}). \tag{31}$$

Combining the first order conditions,

$$\pi(s^t)\beta^t\left(c(s^t) - \bar{c}\right) = (1+r)\sum_{s^{t+1}} \pi(s^{t+1})\beta^{t+1}\left(c(s^{t+1}) - \bar{c}\right) \tag{32}$$

$$c(s^t) = (1+r)\beta \sum_{s^{t+1}} \pi(s^{t+1}|s^t)c(s^{t+1}). \tag{33}$$

With $(1+r)\beta = 1$, consumption is a random walk,

$$c(s^t) = \mathbb{E}_t\, c(s^{t+1}) \tag{34}$$

As in the deterministic model, we can iterate on the budget constraint. After k iterations we have

$$b(s^t) = \frac{b(s^{t+k})}{(1+r)^k} - \sum_{j=0}^{k} \frac{tb(s^{t+j})}{(1+r)^{j+1}}. \tag{35}$$

Take the time-t expectation and then the limit as $k \to \infty$, to yield

$$b(s^t) = -\mathbb{E}_t \sum_{j=0}^{\infty} \frac{tb(s^{t+j})}{(1+r)^{j+1}}. \tag{36}$$

[This requires that $\lim_{j\to\infty} \mathbb{E}_t \frac{b(s^{t+j})}{(1+r)^{t+j}} = 0$.] The intuition is the same: the value of today's net foreign asset position is equal to the discounted expected value of future trade surpluses. Substitute the definition of the trade balance and rearrange to form (note that I pulled a $1+r$ term out of the summations)

$$b(s^t) = -\frac{1}{1+r}\mathbb{E}_t \sum_{j=0}^{k} \frac{y(s^{t+j})}{(1+r)^j} + \frac{1}{1+r}\mathbb{E}_t \sum_{j=0}^{k} \frac{c(s^{t+j})}{(1+r)^j}. \tag{37}$$

Swap the summation for the expectation and the consumption term is

$$\mathbb{E}_t\, c(s^t) + \frac{\mathbb{E}_t\, c(s^{t+1})}{(1+r)} + \frac{\mathbb{E}_t\, c(s^{t+2})}{(1+r)^2} + \dots \tag{38}$$

[1]Todo: with mean μ in the process, we have $y(s^{t+j}) = \mu\frac{1-\rho^j}{1-\rho} + \rho^j y(s^t)$ and $c(s^t) = \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{\mu}{1-\rho}\frac{1-\rho^j}{(1+r)^j} + \left(\frac{\rho}{1+r}\right)^j y(s^t)\right) + rb(s^t)$. Need to double check and work out.

and since consumption is a random walk (apply iterated expectations), the consumption term is just

$$\frac{1}{1+r}\,\mathbb{E}_t \sum_{j=0}^{k} \frac{c(s^{t+j})}{(1+r)^j} = \frac{1}{r}c(s^t). \tag{39}$$

Substitute this into (37) and solve for $c(s^t)$,

$$rb(s^t) + \frac{r}{1+r}\,\mathbb{E}_t \sum_{j=0}^{k} \frac{y(s^{t+j})}{(1+r)^j} \quad = \quad c(s^t). \tag{40}$$

The household consumes a constant share of its *expected* wealth, which is made up of the its net asset position and the expected value of its endowment.

We still have $y(s^t)$ to deal with. From (27) and the law of iterated expectations, we have $\rho^j y(s^t) = \mathbb{E}_t\, y(s^{t+j})$. Use this to get rid of the sum,

$$c(s^t) \quad = \quad rb(s^t) + \frac{r}{1+r}\sum_{j=0}^{\infty} \frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} \tag{41}$$

$$c(s^t) \quad = \quad rb(s^t) + \frac{r}{1+r}\sum_{j=0}^{\infty} \frac{\rho^j y(s^t)}{(1+r)^j} \tag{42}$$

$$c(s^t) \quad = \quad rb(s^t) + \frac{r}{1+r} y(s^t) \sum_{j=0}^{\infty} \frac{\rho^j}{(1+r)^j} \tag{43}$$

$$c(s^t) \quad = \quad rb(s^t) + y(s^t)\frac{r}{1+r-\rho}. \tag{44}$$

So now we have consumption as a function of the current endowment and last period's net foreign asset position. Since $\rho \in (-1, 1)$, the coefficient on $y(s^t)$ is always less than one. Consumption responds less than output — this is consumption smoothing. We will discuss the dependence on ρ shortly. The other variables of interest are

$$tb(s^t) \quad = \quad y(s^t) - c(s^t) = y(s^t)\frac{1-\rho}{1+r-\rho} - rb(s^t) \tag{45}$$

$$ca(s^t) \quad = \quad tb(s^t) + rb(s^t) = y(s^t)\frac{1-\rho}{1+r-\rho} \tag{46}$$

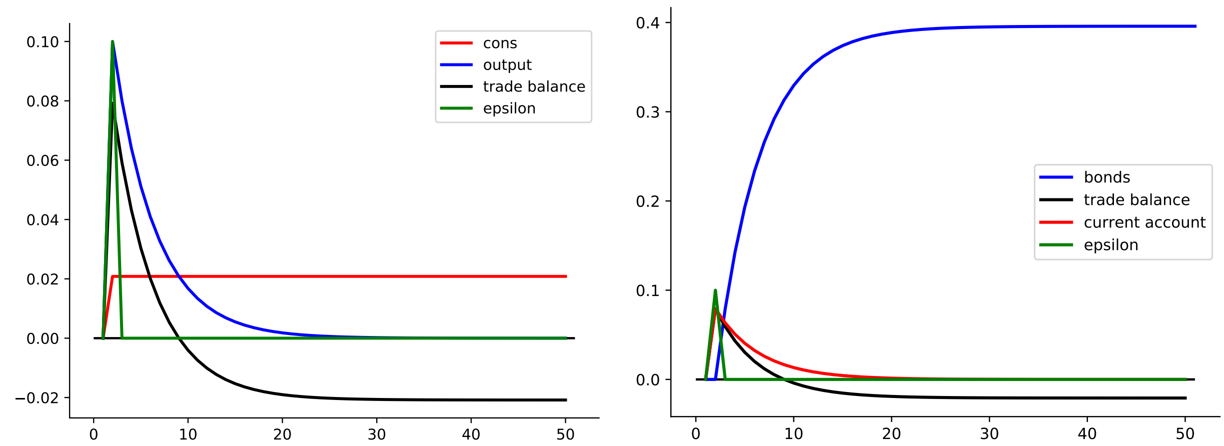
$$b(s^{t+1}) \quad = \quad b(s^t) + ca(s^t) = b(s^t) + y(s^t)\frac{1-\rho}{1+r-\rho}. \tag{47}$$

External debt follows a random walk (the coefficient on $b(s^t)$ is one). The behavior of the model depends on the persistence of the endowment process, which is governed by ρ. Consider an innovation to the endowment, $\epsilon(s^t)$.

1. Temporary shocks, $\rho = 0$. Since $rb(s^t)$ is already determined at time t, consumption changes according to $\Delta c(s^t) = \frac{r}{1+r}\epsilon(s^t)$. This is a purely transitory shock, so consumption does not change much — the shock gets smoothed. The current account absorbs most of the shock,

$$ca(s^t) = \frac{1}{1+r}[0 \times y(s^{t-1}) + \epsilon(s^t)] = \epsilon(s^t)\frac{1}{1+r}, \tag{48}$$

Figure 2: Impulse response in the stochastic endowment model ($r = 0.05$ and $\rho = 0.8$).



(a) Output and consumption

(b) Current account and net foreign assets

and $b(s^{t+1}) = b(s^t) + \epsilon(s^t)\frac{1}{1+r}$. If the shock is negative, the net asset position decreases and the household borrows to smooth consumption. The current account is procyclical and about as volatile as output, $\sigma(c) = \frac{1}{1+r}\sigma(y)$.

2. Almost permanent shocks, $\rho \to 1$. Since $rb(s^t)$ is already determined at time t, consumption changes according to $\Delta c(s^t) = \epsilon(s^t)$. This is a "permanent" change in income, so you consume it. This means that the household does not adjust its savings so $ca_{(}s^t) = 0$ and $tb(s^t) = -rb(s^t)$. The current account is acyclical and is less volatile (duh) than output.

3. Persistent, but not permanent shocks, $\rho \in (0, 1)$. Shocks mean revert. Suppose $\epsilon(s^t) > 0$. Consumption jumps up, but by less than the shock. While output is "high" we have $ca(s^t) > 0$ and the country's net foreign asset position increases. After the shock has decayed away, the country finances the increases consumption through the return on its savings, $\Delta c = r\Delta b$. The impulse response functions are plotted in Figure 2.

The program `soe-random.jl` computes the equilibrium of this model. It is useful to experiment with different values of ρ in the program to get a feel for how the persistence of the output process affects the current account.

1.2.1 Nonstationary endowment

In this section we introduce shocks to the growth rate of the endowment process.

$$\Delta y(s^t) = \rho \Delta y(s^{t-1}) + \epsilon(s^t) \quad (49)$$

$$y(s^t) = y(s^{t-1}) + \rho \Delta y(s^{t-1}) + \epsilon(s^t) \quad (50)$$

where $\epsilon \sim N(0, \sigma_\epsilon^2)$ and $\rho \in [0, 1)$. A feature of this specification is that a one-time shock from ϵ has a permanent effect on the level of future income. To see this, suppose that $\epsilon(s^t) > 0$ and, for all $j = 0, 1, \ldots$, $\epsilon(s^{t+j}) = 0$. For simplicity, assume that $\Delta y(s^{t-1}) = 0$.

$$y(s^{t+1}) = y(s^t) + \rho \Delta y(s^t) + \epsilon(s^{t+1}) \quad (51)$$

since $\epsilon(s^{t+1}) = 0$

$$y(s^{t+1}) = y(s^t) + \rho\Delta y(s^t) \tag{52}$$

and $\Delta y(s^t) = \rho\Delta y(s^{t-1}) + \epsilon(s^t)$

$$y(s^{t+1}) = y(s^t) + \rho\left[\rho\Delta y(s^{t-1}) + \epsilon(s^t)\right] \tag{53}$$

with $\Delta y(s^{t-1}) = 0$

$$y(s^{t+1}) = y(s^t) + \rho\left[\rho + \epsilon(s^t)\right] \tag{54}$$

$y(s^t) = y(s^{t-1}) + \rho\Delta y(s^{t-1}) + \epsilon(s^t)$

$$y(s^{t+1}) = y(s^{t-1}) + \epsilon(s^t) + \rho\epsilon(s^t), \text{ and similarly,} \tag{55}$$

$$y(s^{t+2}) = y(s^{t-1}) + \epsilon(s^t) + \rho\epsilon(s^t) + \rho^2\epsilon(s^t) \tag{56}$$

$$\lim_{j\to\infty} y_{t+j} = y(s^{t-1}) + \frac{1}{1-\rho}\epsilon(s^t) \tag{57}$$

Since $\epsilon(s^t)$ has mean zero, this is also true in expectation.

We haven't changed anything about the environment besides the shock process, so the first order conditions, etc. are the same as in the model with stationary shocks. Like we did before, lets work out the solution to the model, focusing on the behavior of the current account. As before, consumption and the current account are

$$c(s^t) = rb(s^t) + \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} \tag{58}$$

$$ca(s^t) = y(s^t) - c(s^t) + rb(s^t). \tag{59}$$

Substitute consumption into the current account definition, then peel off the $t = 0$ term in the sum and combine it with the $y(s^t)$ term

$$ca(s^t) = y(s^t) - \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} \tag{60}$$

$$ca(s^t) = \frac{1}{1+r}y(s^t) - \frac{r}{1+r}\sum_{j=1}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} \tag{61}$$

break the summation up into

$$ca(s^t) = \frac{1}{1+r}y(s^t) - \sum_{j=1}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} + \frac{1}{1+r}\sum_{j=1}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} \tag{62}$$

$$ca(s^t) = -\sum_{j=1}^{\infty}\frac{\mathbb{E}_t\, y(^{t+j})}{(1+r)^j} + \frac{1}{1+r}\sum_{j=0}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} \tag{63}$$

$$ca(s^t) = -\sum_{j=1}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j})}{(1+r)^j} + \sum_{j=1}^{\infty}\frac{\mathbb{E}_t\, y(s^{t+j-1})}{(1+r)^j} \tag{64}$$

$$ca(s^t) = -\sum_{j=1}^{\infty}\frac{\mathbb{E}_t\, \Delta y(s^{t+j})}{(1+r)^j}. \tag{65}$$

Given our specification of the endowment, we have $\mathbb{E}_t\, \Delta y(s^{t+j}) = \rho^j \Delta y(s^t)$. Use in the above equation to yield

$$
\begin{aligned}
ca(s^t) &= -\sum_{j=1}^{\infty} \frac{\rho^j \Delta y(s^t)}{(1+r)^j} && (66)\\
ca(s^t) &= -\Delta y(s^t) \sum_{j=1}^{\infty} \left(\frac{\rho}{1+r}\right)^j && (67)\\
ca(s^t) &= -\frac{\rho}{1+r-\rho} \Delta y(s^t). && (68)
\end{aligned}
$$

Compare this to the current account behavior in the model with stationary shocks (46). The signs are flipped — this model is qualitatively different than the model that is identical in every way except for the shock process. The intuition is simple. Suppose the economy is hit with a positive innovation to the endowment

- In the model with a stationary output process, the shock means that output will be **temporarily** higher. The optimal thing to do is save some of it (how much depends on the persistence of the process) and spread the temporary increase across time.

$$ca(s^t) = y(s^t)\frac{1-\rho}{1+r-\rho}$$

- In the model with a nonstationary output process, the innovation means that output will be permanently higher. The optimal thing to do is to borrow today against the future income increase. Again, this depends on ρ.

$$ca(s^t) = -\frac{\rho}{1+r-\rho}\Delta y(s^t)$$

We have the current account, so now work out the consumption process. Start by differencing the definition of the current account, then solve for consumption growth.

$$
\begin{aligned}
ca(s^t) - ca(s^{t-1}) &= \Delta y(s^t) - \Delta c(s^t) + r\left[b(s^t) - b(s^{t-1})\right] && (69)\\
\Delta c(s^t) &= \Delta y(s^t) + r\left[b(s^t) - b(s^{t-1})\right] - \left[ca(s^t) - ca(s^{t-1})\right] && (70)
\end{aligned}
$$

Now take another definition of the current account $ca(s^{t-1}) = b(s^t) - b(s^{t-1})$ and write

$$
\begin{aligned}
\Delta c(s^t) &= \Delta y(s^t) + r\left[ca(s^{t-1})\right] - \left[ca(s^t) - ca(s^{t-1})\right] && (71)\\
\Delta c(s^t) &= \Delta y(s^t) - ca(s^t) + (1+r)ca(s^{t-1}) && (72)
\end{aligned}
$$

substitute the solutions for the current accounts

$$
\begin{aligned}
\Delta c(s^t) &= \Delta y(s^t) + \frac{\rho}{1+r-\rho}\Delta y(s^t) - \frac{(1+r)\rho}{1+r-\rho}\Delta y(s^{t-1}) && (73)\\
\Delta c(s^t) &= \frac{1+r}{1+r-\rho}\Delta y(s^t) - \frac{(1+r)\rho}{1+r-\rho}\Delta y(s^{t-1}) && (74)\\
\Delta c(s^t) &= \frac{1+r}{1+r-\rho}\left[\rho\Delta y(s^{t-1}) + \epsilon(s^t)\right] - \frac{(1+r)\rho}{1+r-\rho}\Delta y(s^{t-1}) && (75)\\
\Delta c(s^t) &= \frac{1+r}{1+r-\rho}\epsilon(s^t) && (76)
\end{aligned}
$$

How does the behavior of the economy change with ρ?

1. When $\rho = 0$, the income level is a random walk and income and consumption move one-for-one, $\Delta c(s^t) = \epsilon(s^t)$ (see equation 76) and the current account doesn't change, $ca(s^t) = 0$. This is the same as the case in which $\rho \to 1$ in the stationary income model.
2. When $\rho > 0$, income is stationary in first-differences. Consumption growth rates move more than one-for-one with income growth rates. Suppose $r = 0.04$ and $\rho = 0.5$. Then $\Delta c = 1.93\epsilon$

When turning to the data, we will often look at second moments. (The data are usually made stationary through some kind of filter.) In this model we have[2]

$$\sigma(\Delta c) = \frac{1+r}{1+r-\rho}\sigma(\epsilon) \tag{77}$$

$$\sigma(\Delta c) = \frac{1+r}{1+r-\rho}\sqrt{1-\rho^2}\sigma(\Delta y) \tag{78}$$

where the second line follows from the ϵ shocks being normal: $\sigma^2(\Delta y) = \frac{\sigma^2(\epsilon)}{1-\rho^2}$.

1.3 Data

The classic approach to business cycle analysis is to use the first moments of the data to calibrate or estimate the model and to compare the model's second moments to the second moments in the data. If the model can replicate the second moments in the data, the model is considered successful. In general, the idea is to use one set of moments (first or otherwise) to parameterize the model and a different set of moments to get a sense of model fit.

Business cycle data — international or otherwise — is typically made stationary before computing second moments. There are several ways to "detrend" the data by extracting a cyclical component y_t^c and a smoothed component, or a trend component, y_t^s.

1. First differences. $y_t^c = y_t - y_{t-1}$. Simple. Consistent with model in section 1.2.1.
2. The Hodrick-Prescott filter. Very popular. A smoothing parameter controls the frequency of the extracted cycle. Not great if end points are not on trend. Let $y_t = y_t^s + y_t^c$.

$$\min_{y_t^s} \sum_{t=1}^{T} (y_t^s - y_t)^2 + \lambda \sum_{t=2}^{T} \left[\left(y_{t+1}^s - y_t^s\right) - \left(y_t^s - y_{t-1}^s\right)\right]^2 \tag{79}$$

   The smoothing parameter λ, determines the amount of nonlinearity in the trend component. As $\lambda \to \infty$, the trend is linear. If $\lambda = 0$ the trend is the data. Hodrick and Prescott liked $\lambda = 1600$ for quarterly data and $\lambda = 100$ for annual data. Ravn and Uhlig (2002) suggest 6.25 for annual data.
3. Band-pass filters. Taken from signal processing. It passes business cycles of certain frequencies and rejects longer or shorter business cycles.
4. Regressions on time trends.

[2] Reminder (for me, mostly): $\sigma^2(y_t) = \rho^2\sigma^2(y_{t-1}) + \sigma^2(\epsilon_2)$ means that $\sigma(y) = \sigma(\epsilon)/\sqrt{1-\rho^2}$.

First differences and the HP filter are popular. Band-pass filters used to be more popular. People argue a lot about how to detrend data and much has been said about the HP filter. Despite papers such as "Why you should never use the Hodrick-Prescott filter," it remains a very popular way to approach the data.

The standard operating procedure for generating business cycle moments using the HP filter:

1. Take logs of the data: $\{y_t\} \rightarrow \{\log y_t\}$
2. Apply the HP filter using your favorite statistical package to yield $\{y_t^s, y_t^c\}$. Note that since the data are in logs, the cyclical component is a percent deviation from the trend component.
3. Compute moments from $\{y_t^c\}$, e.g., $\sigma\left(y_t^c\right)$

For variables that can be negative (the trade balance, current account) do not take logs. Instead, express them as shares of GDP, then continue at step 2.

Table 1: Business cycle second moments

	$\sigma(y)$	$\sigma\left(\frac{tb}{y}\right)$	$\rho\left(\frac{tb}{y}\right)$	$\frac{\sigma(c)}{\sigma(y)}$
Industrialized	1.34	1.02	–0.17	0.94
Emerging	2.74	3.22	–0.51	1.45

The data are from Aguiar and Gopinath (2007). Note that the trade balance has been used instead of the current account. The two are very similar for most countries, and data on the trade balance is much easier to obtain, especially for emerging economies. How are "rich" and "poor" countries different?

1. Output is more volatile in emerging markets.
2. The trade balance is more volatile and more counter-cyclical in emerging markets.
3. Consumption is more volatile than output in emerging markets.

1.4 Two goods and relative prices

[todo: add s^t notation; clean up; bonds denominated in foreign goods]

1.4.1 Preliminaries

Consumption is made up of two goods, c_m and c_x, the import and export good,

$$c_t = \left(c_{mt}^{\frac{\gamma-1}{\gamma}} + \tau^{\frac{1}{\gamma}} c_{xt}^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}}. \tag{80}$$

The country is endowed with the export good, and its price, p_x, is exogenous and stochastic. Normalize $p_m = 1$. This introduces the concept of the *terms of trade*,

$$tot_t = \frac{p_{xt}}{p_{mt}}, \tag{81}$$

but be careful — some people define the terms of trade as p_m/p_x. The price of a unit of the consumption good solves

$$P_t = \min_{c_x, c_m} \quad p_{xt} c_{xt} + c_{mt} \tag{82}$$

$$\text{s.t. } 1 \quad = \left(c_{mt}^{\frac{\gamma-1}{\gamma}} + \tau^{\frac{1}{\gamma}} c_{xt}^{\frac{\gamma-1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}} \tag{83}$$

which is

$$P_t = \left(1 + \tau p_{xt}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} . \tag{84}$$

Now we have potentially differing consumption prices in the home country and the rest of the world [e.g., suppose the weights on m and x goods differ in the consumption aggregator]. Let P^* be the price of consumption in the rest of the world and e be the foreign currency cost of one unit of the home currency. The *real exchange rate* is defined as

$$rer_t \quad = \quad e_t \frac{P_t}{P_t^*}. \qquad \text{In what units is the rer?} \tag{85}$$

$$\text{rer units} \quad = \quad \frac{\text{dollars}}{\text{peso}} \frac{\text{pesos / home basket}}{\text{dollars/ row basket}} = \frac{\text{row basket}}{\text{home basket}} \tag{86}$$

An increase in the real exchange rate is an appreciation of the home currency and a depreciation of the foreign currency — it now takes more foreign baskets to purchase one home basket.

The bond is denominated in units of SOE consumption basket. The household budget constraint at time t is

$$p_{xt} c_{xt} + c_{mt} + P_t b_{t+1} \quad = \quad p_{xt} y_t + (1+r) P_t b_t \qquad \text{or} \tag{87}$$

$$P_t c_t + P_t b_{t+1} \quad = \quad p_{xt} y_t + (1+r) P_t b_t \tag{88}$$

$$c_t + b_{t+1} \quad = \quad \frac{p_{xt}}{P_t} y_t + (1+r) b_t \tag{89}$$

The nominal trade balance is

$$tb_t \quad = \quad p_{x,t}(y_t - c_{x,t}) - c_{m,t} = p_{x,t} y_t - P_t c_t \tag{90}$$

where the last line follows because $p_x c_x + p_m c_m = Pc$. This is the nominal trade balance because it has fluctuating prices in it. The current account is

$$b_{t+1} - b_t \quad = \quad \frac{p_{xt}}{P_t} y_t - c_t + r b_t \tag{91}$$

$$b_{t+1} - b_t \quad = \quad \frac{tb_t}{P_t} + r b_t \tag{92}$$

Notice that the trade balance is expressed in the units of consumption goods that it can provide. This makes sense, since the bonds are denominated in units of the home consumption basket.

1.4.2 Model

$$\max_{c(s^t),b(s^{t+1})} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \pi(s^t)\beta^t u(c(s^t)) \tag{93}$$

$$\text{s.t. } c_t + b_{t+1} = \frac{p_{xt}}{P_t} y_t + (1+r)b_t \qquad \forall s^t \tag{94}$$

along with the usual no-Ponzi, transversality, etc. In this case, fluctuations in the quantity of the endowment and fluctuations in the **value of the endowment in terms of consumption**, p_x/P generate income fluctuations.

Special case

Assume that the exported good is not consumed, only the imported good is consumed. Since $p_m(s^t) = 1$ then $P(s^t) = 1$. Assume that the endowment is constant $y(s^t) = 1$ and that $(1+r)\beta = 1$. The utility function is $u(c) = -(c_t - \bar{c})^2 \frac{1}{2}$. Further, assume that

$$tot(s^t) = \rho\, tot(s^{t-1}) + \epsilon(s^t) \tag{95}$$

Now the country is responding to terms-of-trade shocks that change its income. The we have, as before,

$$b(s^t) = -\mathbb{E}_t \sum_{j=0}^{\infty} \frac{tot(s^{t+j}) - c(s^{t+j})}{(1+r)^{j+1}} \tag{96}$$

All of our previous analysis goes through as before. And the relationship between the current account and the terms of trade is

$$ca(s^t) = \frac{1-\rho}{1+r-\rho} tot(s^t) \tag{97}$$

This relationship implies that an increase in the terms of trade should lead to an increase in the current account. As before, the more persistent is the shock, the less the impact should be on the current account. This relationship was studied in Obstfeld (1982) and Svensson and Razin (1983). [USG book covers this in more detail in section 7.3.2.]

2 Production

Production models will generally be too difficult to solve analytically. The usual way forward is to use local approximations to the model equilibrium, which raises the question: "Local approximation around what?" A natural point would be the steady state, but our models do not have a steady state that is not history dependent. Lets start with a deterministic model.

$$\max_{c,b_{t+1},i_t,k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{98}$$

$$\text{s.t. } c_t + b_{t+1} + i_t = z_t k_t^{\alpha} + (1+r)b_t \tag{99}$$

$$k_{t+1} = k_t(1-\delta) + i_t \tag{100}$$

Here we have introduced capital k and investment i. The production function is $y = zk^\alpha$ and z will vary over time in a deterministic way. Let λ_t be the multiplier on the budget constraint and μ_t be the multiplier on the law of motion for capital. [The multiplier μ is the benefit from a marginal increase in the capital stock. Maybe a better notation is q.] The first order conditions are

$$c_t: \quad u'(c_t)\beta^t = \lambda_t \tag{101}$$
$$i_t: \quad \lambda_t = \mu_t \tag{102}$$
$$b_{t+1}: \quad \lambda_t = (1+r)\lambda_{t+1} \tag{103}$$
$$k_{t+1}: \quad \mu_t = (1-\delta)u_{t+1} + \alpha z_{t+1} k_{t+1}^{\alpha-1}\lambda_{t+1} \tag{104}$$

Combine the first order conditions to yield

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{105}$$
$$\frac{\lambda_t}{\lambda_{t+1}} = (1+r) = (1-\delta) + \alpha z_{t+1} k_{t+1}^{\alpha}. \tag{106}$$

The first equation says that the change in the marginal utility of consumption is related to $(1+r)\beta$ and the second says that the return to capital must equal the interest rate. These two equations plus the budget constraint the LOM for capital and the transversality condition characterize equilibrium.

Do the usual budget constraint math and impose transversality to yield

$$c_t = \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{z_{t+j}k_{t+j}^{\alpha} - i_{t+j}}{(1+r)^j} + rb_t. \tag{107}$$

Consumption is now weighted average of output net of investment costs plus capital income. If we can characterize net output, the rest of the model works as in the endowment economies.

2.1 Steady state

We are looking for a "steady state" in which $k_t = k_{ss}$, $c_t = c_{ss}$, $i_t = i_{ss}$ and $b_t = b_{ss}$ when we impose that $z_t = z_{ss}$. The LOM of capital says $i_{ss} = \delta k_{ss}$. The bonds Euler equation says that $(1+r)\beta = 1$ in a steady state (not a balanced growth path) so that $c_t = c_{t+1}$. Euler for capital says

$$r + \delta = \alpha z_{ss} k_{ss}^{\alpha-1} \tag{108}$$
$$\left(\frac{\alpha z_{ss}}{r+\delta}\right)^{\frac{1}{1-\alpha}} = k_{ss} \tag{109}$$

Great, so capital is pinned down. Note that steady state capital is increasing in α and z, and decreasing in δ and r. [Why?] What about c_{ss} and b_{ss}? Before turning to those, let's look at a simple example.

One time, permanent change; no depreciation

Let's assume $z_t = z$ until time 0 and then unexpectedly changes to $z_t = z'$ for ever after. Let $\delta = 0$, $b_{s0} = 0$, and $\beta(1+r) = 1$. At time -1, the economy is in a steady state. Now let's think about what happens as z changes. We start with a simple special case.

var	$t=-1$	$t=0$	$t=1$	$t=2+$
z	z	z'	z'	z'
k	$k_{s0} = (\frac{\alpha z}{r})^{\frac{1}{1-\alpha}}$	$k_{s0} = (\frac{\alpha z}{r})^{\frac{1}{1-\alpha}}$	$k_{s1} = (\frac{\alpha z'}{r})^{\frac{1}{1-\alpha}}$	$k_{s1} = (\frac{\alpha z'}{r})^{\frac{1}{1-\alpha}}$
y	$y_{s0} = zk_{s0}^{\alpha}$	$z'k_{s0}^{\alpha}$	$y_{s1} = z'k_{s1}^{\alpha}$	$y_{s1} = z'k_{s1}^{\alpha}$
i	0	$k_{s1} - k_{s0}$	0	0
c	$c_{s0} = y_{s0}$	$c_0 = \frac{r}{1+r}(z'k_{s0}^{\alpha} - i_0) + \frac{1}{1+r}y_{s1}$	$c_{s1} = y_{s1} + rb_{s1}$	$c_{s1} = y_{s1} + rb_{s1}$
tb	0	$tb_0 = z'k_{s0}^{\alpha} - c_0 - i_0 < 0$	$-rb_{s1} > 0$	$-rb_{s1} > 0$
b	$b_{s0} = 0$	$b_{s0} = 0$	$b_{s1} = tb_0$	$b_{s1} = tb_0$
ca	0	tb_0	0	0

Notice that $c_0 = c_{s1}$: the economy jumps to the new steady state level of consumption, even though output does not. This difference is financed from abroad ($tb_0 < 0$) and the country pays for it in the future with $-rb_{s1} = -rtb_0 > 0$ in perpetuity.[a] [How would this look different in a closed economy?]

Notice that the trade balance at time 0 is being used to finance both an increase in capital and an increase in consumption. The capital stock increases because it is now more productive, and r_k must equal r. [Todo: prove that $c_0 > c_{s0}$.]

[a] Figure 3.1 in USG book is a nice visualization of this.

In our simple example, computing the new steady state debt level required knowing what happened along the transition path. The new steady state is history dependent. We can see this more generally be continuing to solve for the steady state. We have already pinned down the steady state capital stock. Now, consider the budget constraint.

$$c_{ss} + b_{ss} + i_{ss} = k_{ss}^{\alpha} + (1+r)b_{ss} \tag{110}$$

$$c_{ss} + \delta k_{ss} = k_{ss}^{\alpha} + rb_{ss} \tag{111}$$

$$c_{ss} = \left(\frac{\alpha z_{ss}}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha z_{ss}}{r+\delta}\right)^{\frac{1}{1-\alpha}} + rb_{ss} \tag{112}$$

What is the value of consumption? Depends on the value of debt. We have nothing in the model left to pin down a steady state level of debt. [We already knew this: debt follows a random walk.] This does not mean that the steady state is indeterminate. It means that the steady state is history dependent.

In a general equilibrium model, the interest rate would be endogenous and we would not have this problem. This suggests ways to impose stationarity. There are several ways to make this model stationary.

2.2 Stationarity

Let us assume that r^* is the steady-state world interest rate and the country pays a premium when debt gets too large,

$$r = r^* + \psi\left(e^{\bar{b}-b_t} - 1\right) \tag{113}$$

so that the interest increases as debt becomes more negative than $\bar{b}$ and $r = r^*$ when debt is equal to $\bar{b}$. Now (105) is

$$u'(c_{ss}) = \beta(1 + r^* + \psi\left(e^{\bar{b}-b_{ss}} - 1\right))u'(c_{ss}) \tag{114}$$

$$1 = \beta(1 + r^* + \psi\left(e^{\bar{b}-b_{ss}} - 1\right)) \tag{115}$$

$$1 - (1+r)\beta = \beta\psi\left(e^{\bar{b}-b_{ss}} - 1\right) \tag{116}$$

$$0 = e^{\bar{b}-b_{ss}} - 1 \tag{117}$$

so $b_{ss} = \bar{b}$ and $r = r^*$. Now the debt level is not history dependent. Now k_{ss} is as before but with r^* and consumption is

$$c_{ss} = \left(\frac{\alpha}{r^* + \delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha}{r^* + \delta}\right)^{\frac{1}{1-\alpha}} + r\bar{b}. \tag{118}$$

2.3 Linear solutions

[This discussion follows USG 4.6. Reading that will make these notes make more sense, I hope.]

We often linearize with respect to the logs of some variables and the levels of others. Using logs is convenient when the empirical counterpart is measured as a percent deviation — what you would have after HP filtering, for example. Variables that can be negative, or variables already expressed in rates are linearized in levels. In our example, we will use logs for c_t, k_t, and z_t and levels for r_t and b_t.

Let's linearize the following function

$$s_t = \mathbb{E}_t\, m(u_t, v_t, z_{t+1}) \tag{119}$$

where u_t and z_t are log-able and v_t can take negative values. Log deviations as

$$\hat{s}_t = \log\left(\frac{s_t}{s_{ss}}\right) \approxeq \frac{s_t}{s_{ss}} - 1 \tag{120}$$

so that $\hat{s}_t$ is a percent deviation as long as s_t is close to its steady state value. This also means that $s_t = e^{\hat{s}_t} s_{ss}$, which we will use in a moment. For variables in levels, denote $\hat{v}_t = v_t - v_{ss}$, so that $\hat{v}_t + v_{ss} = v_t$. A first-order approximation to $y = f(x)$ in logs is

$$\begin{aligned}
\log(y_t) &= \log(f(x_t)) && (121)\\
\log(y_{ss}) + \frac{1}{y_{ss}}(y_t - y_{ss}) &= \log(f(x_{ss})) + \frac{1}{f(x_{ss})} f_x(x_{ss})(x_t - x_{ss}) && (122)\\
\frac{1}{y_{ss}}(y_t - y_{ss}) &= \frac{1}{f(x_{ss})} f_x(x_{ss})(x_t - x_{ss}) && (123)\\
(y_t - y_{ss}) &= f_x(x_{ss})(x_t - x_{ss}) && (124)\\
y_{ss}\hat{y}_t &= f_x(x_{ss})x_{ss}\hat{x}_t && (125)
\end{aligned}$$

This generalizes to a multivariate setting,

$$\begin{aligned}
s_{ss}\hat{s}_t &= m_u(u_{ss}, v_{ss}, z_{ss})u_{ss}\hat{u}_t + m_v(u_{ss}, v_{ss}, z_{ss})\hat{v}_t + m_z(u_{ss}, v_{ss}, z_{ss})z_{ss}\, \mathbb{E}_t\, \hat{z}_{t+1} && (126)\\
\hat{s}_t &= \frac{m_u(u_{ss}, v_{ss}, z_{ss})u_{ss}}{s_{ss}}\hat{u}_t + \frac{m_v(u_{ss}, v_{ss}, z_{ss})}{s_{ss}}\hat{v}_t + \frac{m_z(u_{ss}, v_{ss}, z_{ss})z_{ss}}{s_{ss}}\, \mathbb{E}_t\, \hat{z}_{t+1} && (127)
\end{aligned}$$

where the fraction terms are either elasticities or semi-elasticities.

2.3.1 Linearized equilibrium conditions

The Euler equation for bonds

$$\begin{aligned}
u_c(c_t) - \beta[1 + r^* + p(b_{t+1})]\, \mathbb{E}_t\, u_c(c_{t+1}) &= 0 && (128)\\
u_{cc}c\hat{c}_t - \beta[1 + r^* + p(b)]\, \mathbb{E}_t\, u_{cc}c\hat{c}_{t+1} - \beta p_b(b)u_c\hat{b}_{t+1} &= 0 && (129)\\
\frac{u_{cc}c}{u_c}\hat{c}_t - \frac{u_{cc}c}{u_c}\, \mathbb{E}_t\, \hat{c}_{t+1} - \beta p_b(b)\hat{b}_{t+1} &= 0 && (130)
\end{aligned}$$

note that $p(b) = 0$ and $(1 + r)\beta = 1$.

The Euler equation for capital

$$\begin{aligned}
u_c(c_t) - \beta\, \mathbb{E}_t\{[1 - \delta + z_{t+1}\alpha k_{t+1}^{\alpha-1}]u_c(c_{t+1})\} &= 0\\
u_{cc}c\hat{c}_t - \beta u_c\alpha k^{\alpha-1}z\, \mathbb{E}_t\{\hat{z}_{t+1}\} - \beta u_c z\alpha(\alpha - 1)k^{\alpha-2}k\, \mathbb{E}_t\{\hat{k}_{t+1}\} - \beta u_{cc}[1 - \delta + z\alpha k^{\alpha-1}]c\, \mathbb{E}_t\, \hat{c}_{t+1} &= 0\\
\frac{u_{cc}c}{u_c}\hat{c}_t - \beta\alpha k^{\alpha-1}z\, \mathbb{E}_t\{\hat{z}_{t+1}\} - \beta z\alpha(\alpha - 1)k^{\alpha-2}k\, \mathbb{E}_t\{\hat{k}_{t+1}\} - \beta\frac{u_{cc}c}{u_c}[1 - \delta + z\alpha k^{\alpha-1}]\, \mathbb{E}_t\, \hat{c}_{t+1} &= 0\\
\frac{u_{cc}c}{u_c}\hat{c}_t - \beta\alpha k^{\alpha-1}z[\mathbb{E}_t\{\hat{z}_{t+1}\} + (\alpha - 1)\, \mathbb{E}_t\{\hat{k}_{t+1}\}] + \frac{u_{cc}c}{u_c}[1 - \delta + z\alpha k^{\alpha-1}]\, \mathbb{E}_t\, \hat{c}_{t+1} &= 0\\
\frac{u_{cc}c}{u_c}\hat{c}_t - \frac{u_{cc}c}{u_c}\, \mathbb{E}_t\, \hat{c}_{t+1} - \frac{\beta(1 - \delta) - 1}{\beta(1 - \delta)}[\mathbb{E}_t\{\hat{z}_{t+1}\} + (\alpha - 1)\, \mathbb{E}_t\{\hat{k}_{t+1}\}] &= 0
\end{aligned}$$

note that $1/\beta = 1 - \delta + zk^{\alpha-1}\alpha$, which simplifies that last two lines.

The budget constraint

$$\begin{aligned}
c_t + k_{t+1} - (1 - \delta)k_t + b_{t+1} - z_t k_t^\alpha - [1 + r^* + p(b_t)]b_t &= 0 && (131)\\
c\hat{c}_t + k\hat{k}_{t+1} - [1 - \delta + \alpha zk^{\alpha-1}]k\hat{k}_t + \hat{b}_{t+1} - zk^\alpha z\hat{z}_t - [1 + r^* + p(b) + bp_b(b)]\hat{b}_t &= 0 && (132)\\
\frac{c}{y}\hat{c}_t + \frac{k}{y}\hat{k}_{t+1} - [1 - \delta + \alpha zk^{\alpha-1}]\frac{k}{y}\hat{k}_t + \frac{1}{y}\hat{b}_{t+1} - z\hat{z}_t - [1 + r^* + bp_b(b)]\hat{b}_t &= 0 && (133)
\end{aligned}$$

The law of motion for productivity

$$\begin{aligned}
\mathbb{E}_t \log(z_{t+1}) - \rho \log(z_t) &= 0 && (134)\\
\mathbb{E}_t\, \hat{z}_{t+1} - \rho\hat{z}_t &= 0 && (135)
\end{aligned}$$

2.3.2 Solving

Let $\hat{x}_t = [\hat{b}_t, \hat{k}_t, \hat{z}_t]'$ and $\hat{y}_t = [\hat{c}_t]'$. We are looking for solutions of the form

$$\hat{x}_{t+1} = h\hat{x}_t + \eta\epsilon_{t+1} \tag{136}$$

$$\hat{y}_{t+1} = g\hat{y}_t \tag{137}$$

where h and g are matrices with coefficients to be determined. In theory, substitute these rules into the linearized equations and solve for g and h. In practice, this requires some work. See Klein (2000), Uhlig (1999) and USG Appendix 4.14 for details. The issue is that we need solutions in which $\hat{x}_{t+j} \to 0$ and $\hat{y}_{t+j} \to 0$ as $j \to \infty$. The model hast to return to the steady state. [see why we needed a stationary model?]

2.3.3 Impulse responses

A useful tool for understanding the model is the impulse response function. At time zero, an impulse of (typically) one standard deviation to ϵ_0 occurs and $\epsilon_t = 0$ forever after. The impulse response traces out the expected value of the model following the impulse. For variable w, the impulse response is

$$IR(\hat{w}_t) = E_0\hat{w}_t - E_{-1}\hat{w}_t \tag{138}$$

and note that for our purposes $E_{-1}\hat{w}_t = w_{ss}$. We can compute the impulse response functions directly from the solution to the linearized model. For the state variables, this works out to be $IR(\hat{x}_t) = h^t x_{ss}$ and for the controls, $IR(\hat{y}_t) = gh^t x_{ss}$.

2.3.4 Dynare

Dynare solves several headaches. You input the non-linearized equations that characterize equilibrium and it takes the derivatives and solves the model for you — assuming a solution exists!

Dynare uses different timing conventions than I have used in these notes.