Constant Elasticity of Substitution Demand

Constant elasticity of substitution preferences will show up often in international trade; they are a simple way to aggregate over many goods. Consider the case with a continuum of goods.

$$\max \; U(\mathbf{c}) = \left(\int_0^1 b(i)c(i)^\rho di\right)^{1/\rho} \tag{1}$$
$$\text{s.t.} \; \int_0^1 p(i)c(i)di = I$$

The first order conditions are

$$\left(\int_0^1 b(i)c(i)^\rho di\right)^{1/\rho-1} b(i')c(i')^{\rho-1} = \lambda p(i') \qquad i' \in [0,1] \tag{2}$$

multiply each FOC by $c(i')$ and integrate over them all

$$\left(\int_0^1 b(i)c(i)^\rho di\right)^{1/\rho-1} b(i')c(i')^{\rho} = \lambda p(i')c(i')$$
$$\left(\int_0^1 b(i)c(i)^\rho di\right)^{1/\rho-1} \int_0^1 b(i')c(i')^{\rho}di' = \lambda \int_0^1 p(i')c(i')di'$$
$$\left(\int_0^1 b(i)c(i)^\rho di\right)^{1/\rho} = \lambda \int_0^1 p(i')c(i')di' \tag{3}$$

It is typical to define a unit of utility as

$$C = \left(\int_0^1 b(i)c(i)^\rho di\right)^{1/\rho}. \tag{4}$$

Using this definition, and the budget constraint, we can write (3) as

$$C\lambda^{-1} = I, \tag{5}$$

where $\lambda^{-1} = P$ has the natural interpretation as being the price of one unit of utility. To solve for P, return to (2) and solve for $c(i')^\rho$ and multiply by b

$$c(i')^\rho = \lambda^{\rho/(\rho-1)} b(i')^{-\rho/(\rho-1)} p(i')^{\rho/(\rho-1)} \int_0^1 b(i)c(i)^\rho di \tag{6}$$
$$b(i')c(i')^\rho = \lambda^{\rho/(\rho-1)} b(i')^{-1/(\rho-1)} p(i')^{\rho/(\rho-1)} \int_0^1 b(i)c(i)^\rho di \tag{7}$$

Now integrate all of the FOCs again:

$$\int_0^1 b(\imath')c(\imath')^\rho d\imath' = \lambda^{\rho/(\rho-1)} \int_0^1 b(\imath')^{-1/(\rho-1)} p(\imath')^{\rho/(\rho-1)} d\imath' \int_0^1 b(i)c(i)^\rho di \tag{8}$$

to get

$$P = \lambda^{-1} = \left(\int_0^1 b(\imath')^{\frac{-1}{\rho-1}} p(\imath')^{\frac{\rho}{\rho-1}} d\imath' \right)^{\frac{\rho-1}{\rho}} \tag{9}$$

Properties

Elasticity of Substitution

From (2) we have

$$\frac{c(j)}{c(i)} = \left(\frac{b(j)}{b(i)} \right)^{-\frac{1}{1-\rho}} \left(\frac{p(j)}{p(i)} \right)^{-\frac{1}{1-\rho}} \tag{10}$$

Taking the log of this equation and differentiating with respect to $p(j)/p(i)$ shows that the elasticity of substitution between goods is $\sigma = 1/(1-\rho)$.

Demand Function and Own Price Elasticity

If we substitute P into (2) we have the demand function

$$c(i) = b(i)^{\frac{1}{1-\rho}} \left(\frac{p(i)}{P} \right)^{\frac{-1}{1-\rho}} C, \tag{11}$$

where $\sigma = 1/(1-\rho)$ is the own price elasticity of good i. Note that this is only the case when good i is so small it has a negligible impact on P. This is true when there are a continuum of goods, and is also typically assumed even when the number of goods is finite. A typical assumption is that "the number of goods is large enough that no single good can influence the aggregate price level."

The demand function can be written in expenditure form, as well

$$p(i)c(i) = b(i)^{\frac{1}{1-\rho}} \left(\frac{p(i)}{P} \right)^{\frac{-\rho}{1-\rho}} PC. \tag{12}$$

Limiting Cases

As $\rho \to 0$ this is Cobb-Douglas. As $\rho \to 1$ perfect substitutes, as $\rho \to \infty$ Leontief.

Two Stage Budgeting

While some authors will model preferences exactly as (1), it is common to nest a CES group of products into an economy with other goods and specify the consumer's problem as

$$\begin{aligned} &\max\ U\left(c_0, \left(\int_0^1 b(i)c(i)^\rho di\right)^{1/\rho}\right) \\ \text{s.t. }\ &p_0c_0 + \int_0^1 p(i)c(i)di = I \end{aligned} \tag{13}$$

Using the derivations above, we can rewrite this as a "two stage budgeting problem," where the first stage is

$$\begin{aligned} &\max\ U(c_0, C) \\ \text{s.t. }\ &p_0c_0 + PC = I \end{aligned} \tag{14}$$

which determines the expenditure on the nummeraire good and the differentiated goods. Once we know PC we can solve the problem specified in (1) replacing I with PC.

CES Production

You will also find papers in which the i-goods are considered to be *intermediate goods* and the CES aggregator is meant to be a production function (e.g. Ethier 1982). For example, a model might have a feasibility constraint like

$$C_t + I_t \leq \left(\int_0^1 b(i)y(i)^\rho di\right)^{1/\rho} = Y_t, \tag{15}$$

where C is consumption of the final good and I is the amount of final good used for investment. In a model like this, you might be tempted to think that P is the GDP price deflator. It is not. The GDP deflators in the United States and Canada are chain-weighted Fisher indices. Other countries follow different procedures (some use chained Laspeyres indices) but, in any case, none of the countries use a price index that looks like (9).